CHAPTER 15

PARTIAL DIFFERENTIATION

15.1 Introduction to Partial Derivatives

PREREQUISITES

1. Recall the basic rules of one-variable differentiation (Chapters 1 and 2, Sections 5.2, 5.4, and 6.3).
2. Recall the definition of the derivative for one variable (Section 1.3).
3. Recall the definition of continuity with one variable (Section 3.1).

PREREQUISITE QUIZ

1. Differentiate the following:
   (a) $(\sqrt{x} + 1)^2(x^3 + x - 3)$
   (b) $x/(2 + \cos x)^2$
   (c) $\exp(\tan x)$
2. If $f(x) = x^3 + 2$, what is $[df^{-1}(x)/dx]|_{10}$?
3. Use limits to prove that $(d/dx)x^2 = 2x$.
4. State two conditions that must be met for a one-variable function to be called continuous.

GOALS

1. Be able to state the definition of a partial derivative.
2. Be able to compute partial derivatives.

3. Be able to explain in "common terms" what partial derivatives tell you.
4. Be able to explain the relationship between mixed partials.

STUDY HINTS

1. Partial derivatives. The three main notations used to represent partials are f_x, $\partial z/\partial x$, and $\partial f/\partial x$. Differentiation is performed with respect to the variable in the subscript or the "denominator". In this case, all independent variables in the function f are held constant except for x and differentiation is performed as usual.
2. Definition. The definition of partial derivatives is almost the same as the derivative in one-variable calculus. Remember that only x is changing in $\partial f/\partial x$, so $\partial f/\partial x$ is $\lim_{\Delta x\to 0}\{[f(x+\Delta x,y,z) - f(x,y,z)]/\Delta x\}$.
3. Equality of mixed partials. This concept allows you to calculate second or higher partials in any order as long as you differentiate with respect to each variable the specified number of times. You need continuous partial derivatives to apply this rule.
4. Evaluating partials at a given point. Always remember to differentiate completely before substituting given values. With mixed partials, you may substitute for a variable only after you have completed differentiating in that variable.
5. Limits. Example 9 shows that we need to get the same value in all directions (not just along directions parallel to the x- and y-axes) if the limit exists. To show that a limit does not exist, we can often compute values in directions parallel to the coordinate axes and see if they are unequal.

   With the ε-δ definition, we're interested in what happens near a point. In two dimensions, nearness is defined by surrounding a point by a disk, which represents δ. A ball is used in three dimensions.

6. Continuity. As with one variable, the function is continuous wherever the limit equals the function value.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $f_x = y$, so $f_x(1,1) = 1$; $f_y = x$, so $f_y(1,1) = 1$.

5. By the product rule, $f_x = ye^{xy}\sin(x + y) + e^{xy}\cos(x + y)$, so $f_x(0,0) = 0 + \cos 0 = 1$; $f_y = xe^{xy}\sin(x + y) + e^{xy}\cos(x + y)$, so $f_y(0,0) = 1$. Note that, due to the symmetry of the function, f_y could have been found by reversing the roles of x and y .

9. $f_x = yz$, so $f_x(1,1,1) = 1$; $f_y = xz$, so $f_y(1,1,1) = 1$; $f_z = xy$, so $f_z(1,1,1) = 1$.

13. $\partial z/\partial x = 6x$ and $\partial z/\partial y = 4y$.

17. $\partial u/\partial x = [-yz\exp(-xyz)](xy + xz + yz) + [\exp(-xyz)](y + z) = \exp(-xyz) \times [-yz(xy + xz + yz) + (y + z)]$; $\partial u/\partial y = [-xz\exp(-xyz)](xy + xz + yz) + [\exp(-xyz)](x + z) = \exp(-xyz)[-xz(xy + xz + yz) + (x + z)]$; $\partial u/\partial z = [-xy\exp(-xyz)](xy + xz + yz) + [\exp(-xyz)](x + y) = \exp(-xyz) \times [-xy(xy + xz + yz) + (x + y)]$.

21. The partial is $[xe^y(ye^x + 1) - (xe^y - 1)e^x]/(ye^x + 1)^2 = (xye^xe^y - xe^xe^y + xe^y + e^x)/(ye^x + 1)^2$.

25. $f_x = 6x + (2/y^2)\cos(x/y^2) + y^3(-e^x)$, so $f_x(2,3) = 12 + (2/9)\cos(2/9) - 27e^2$.

29. (a) $\partial z/\partial y = (\sin x)e^{-xy}(-x) = -x(\sin x)e^{-xy}$.

(b) Evaluating $\partial z/\partial y$ at the given points, we have $z_y(0,0) = 0$; $z_y(0,\pi/2) = 0$; $z_y(\pi/2,0) = -\pi/2$; $z_y(\pi/2,\pi/2) = -(\pi/2)\exp(-\pi^2/4)$.

33. $g_u(t,u,v) = 1/(t + u + v) - (tv)\sec^2(tuv)$, so $g_u(1,2,3) = 1/6 - 3\sec^2(6)$.

37. $(\partial/\partial s)\exp(stu^2) = (tu^2)\exp(stu^2)$.

41. Using the idea of Example 2(c), we get $f_z = \lim_{\Delta z\to 0}\{[f(x,y,z+\Delta z) - f(x,y,z)]/\Delta z\}$.

45. (a) $R = 1/(1/R_1 + 1/R_2 + 1/R_3)$, so $\partial R/\partial R_1 = -1/(1/R_1 + 1/R_2 + 1/R_3)^2(-1/R_1^2) = 1/(1 + R_1/R_3 + R_1/R_3)^2$.

(b) $(\partial R/\partial R_1)|_{(100,200,300)} = 1/(1 + 1/2 + 1/3)^2 = (11/6)^{-2} = 36/121$, so R is changing $36/121$ times as fast.

49. From Exercise 15, we get $z = 2x/3y + 7x/3$, so $\partial z/\partial x = 2/3y + 7/3$ and $\partial z/\partial y = -2x/3y^2$. Thus, $\partial^2 z/\partial x^2 = (\partial/\partial x)(\partial z/\partial x) = 0$; $\partial^2 z/\partial x\partial y = (\partial/\partial x)(\partial z/\partial y) = (\partial/\partial y)(\partial z/\partial x) = \partial^2 z/\partial y\partial x = -2/3y^2$; $\partial^2 z/\partial y^2 = (\partial/\partial y)(\partial z/\partial y) = 4x/3y^3$.

53. $\partial u/\partial x = [(x^2 + y^2)^2(2y) - 2xy(4x)(x^2 + y^2)]/(x^2 + y^2)^4 = (-6x^2y + 2y^3)/(x^2 + y^2)^3$; $\partial u/\partial y = [(x^2 + y^2)^2(2x) - 2xy(4y)(x^2 + y^2)]/(x^2 + y^2)^4 = (-6xy^2 + 2x^3)/(x^2 + y^2)^3$; so $\partial^2 u/\partial x^2 = [(x^2 + y^2)^3(-12xy) - (-6x^2y + 2y^3)(6x)(x^2 + y^2)^2]/(x^2 + y^2)^6 = 24xy(x^2 - y^2)/(x^2 + y^2)^4$; $\partial^2 u/\partial y\partial x = (\partial/\partial y)(\partial u/\partial x) = [(x^2 + y^2)^3(-6x^2 + 6y^2) - (-6x^2y + 2y^3)(6y) \times (x^2 + y^2)^2]/(x^2 + y^2)^6 = -6(x^4 - 6x^2y^2 + y^4)/(x^2 + y^2)^4$; $\partial^2 u/\partial x\partial y = (\partial/\partial x)(\partial u/\partial y) = [(x^2 + y^2)^3(-6y^2 + 6x^2) - (-6xy^2 + 2x^3)(6x)(x^2 + y^2)^2]/(x^2 + y^2)^6 = -6(x^4 - 6x^2y^2 + y^4)/(x^2 + y^2)^4 = \partial^2 u/\partial y\partial x$; and $\partial^2 u/\partial y^2 = [(x^2 + y^2)^3(-12xy) - (-6xy^2 + 2x^3)(6y)(x^2 + y^2)^2]/(x^2 + y^2)^6 = -24xy(x^2 - y^2)/(x^2 + y^2)^4$.

57. For any given $\varepsilon > 0$, take $\delta = \varepsilon$. Then, for any (x,y) such that $0 < d((x,y),(x_0,y_0)) < \delta$, i.e., $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, we have $|y - y_0| < \delta$. Thus, $|y - y_0| < \varepsilon$, i.e., $|f(x,y) - \ell| < \varepsilon$, also. So $\lim_{(x,y)\to(x_0,y_0)} y = y_0$.

61. Since the denominator doesn't vanish, the function is a "reasonable" one in which the limit may be evaluated by substituting $(x,y) = (2,3)$. Thus, the limit is $52/\sqrt{13} = 4\sqrt{13}$.

65. By direction substitution, $\lim_{(x,y)\to(0,0)} \sin(x,y) = \sin(0) = 0$.

69. If $u = x^3 - 3xy^2$, then $\partial u/\partial x = 3x^2 - 3y^2$ and $\partial^2u/\partial x^2 = 6x$. Also, $\partial u/\partial y = -6xy$ and $\partial^2u/\partial y^2 = -6x$. Therefore, $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 6x - 6x = 0$. This means that $u(x,y)$ is a harmonic function.

73. (a) $(\partial z/\partial x)|_{(5,3)} = (60y - 2x)|_{(5,3)} = 180 - 10 = 170$ units.

(b) $(\partial z/\partial y)|_{(5,3)} = (60x - 8y)|_{(5,3)} = (300 - 24) = 276$ units. This is the marginal productivity of capital, per million dollars invested, at a labor force of 5000 people and investment level of 3 million dollars.

77. (a) From Exercise 76(a), $f_x = y(x^4 + 4x^2y^2 - y^4)/(x^2 + y^2)^2$. Substitute $x = 0$ into f_x giving, for $y \neq 0$, $f_x(0,y) = y(-y^4)/y^4 = -y$.

(b) From Exercise 76(a), $f_y = x(x^4 - 4x^2y^2 - y^4)/(x^2 + y^2)^2$. Substitute $y = 0$ into f_y giving, for $x \neq 0$, $f_y(x,0) = x(x^4)/x^4 = x$.

(c) Using the definition of the derivative, $f_{yx}(0,0) = \lim_{\Delta x\to 0} \{[f_y(\Delta x,0) - f_y(0,0)]/\Delta x\} = \lim_{\Delta x\to 0} [(\Delta x - 0)/\Delta x] = \lim_{\Delta x\to 0} 1 = 1$. Similarly, $f_{xy}(0,0) = \lim_{\Delta y\to 0} \{[f_x(0,\Delta y) - f_x(0,0)]/\Delta y\} = \lim_{\Delta y\to 0} [(-\Delta y - 0)/\Delta y] = \lim_{\Delta y\to 0} (-1) = -1$.

(d) By the equality of mixed partials, in order to have $f_{xy} = f_{yx}$, both f_{xx} and f_{yy} must be continuous. Without actually differentiating f_x or f_y (from Exercise 76(a)), we can see that each will have $(x^2 + y^2)^3$ in the denominator. Therefore, neither is defined, nor continuous, at $(0,0)$, so the equality of mixed partials does not apply.

SECTION QUIZ

1. Fill in the blanks: Suppose that f is a function of x, y, and z. f_x is the notation used to indicate differentiation with respect to ______ while the variable(s) ______ is (are) held constant.

2. If f_y is a partial derivative, write f_{yy} in Leibniz notation, and explain how to find f_{yy} from f_y.

3. (a) Write $\partial g/\partial z$ for a function $g(x,y,z)$ as a limit.

   (b) Using the limit method, find $g_z(0,1,1)$ for $g(x,y,z) = x^2yz + xz^2 - xy$.

4. Find $\partial^3 g/\partial x^2 \partial y$ for $g(x,y) = \cos(e^x y)$.

5. (a) For the function in Question 4, does $\partial^3 g/\partial x^2 \partial y = \partial^3 g/\partial x \partial y \partial x$?

   (b) In general, does $\partial^3 g/\partial x^2 \partial y = \partial^3 g/\partial x \partial y \partial x$ for any $g(x,y)$? Explain.

6. A careless barber cuts m percent more hair than his customers request according to the following formula: $m = (g + h)/d$, where g is the minutes the barber spends gossipping with the customer; h is the ounces of hair on the customer's head; and d is the expected tip to be received in dollars.

   (a) Compute $\partial m/\partial d$.

   (b) Suppose $\partial m/\partial g = 7$. Explain the meaning of the number.

   (c) What will happen to the customer's hair if the barber expects no tip?

ANSWERS TO PREREQUISITE QUIZ

1. (a) $(\sqrt{x} + 1)[(x^3 + x - 3)/\sqrt{x} + (\sqrt{x} + 1)(3x^2 + 1)]$

   (b) $(2 + \cos x + 2x \sin x)/(2 + \cos x)^3$

   (c) $(\sec^2 x)\exp(\tan x)$

2. 1/300

3. $(d/dx)x^2 = \lim_{\Delta x\to 0} \{[(x + \Delta x)^2 - x^2]/\Delta x\} = \lim_{\Delta x\to 0} [(2x\Delta x + (\Delta x)^2)/\Delta x] =$
$\lim_{\Delta x\to 0} (2x + \Delta x) = 2x$.

4. (i) $\lim_{x\to a} f(x)$ exists and (ii) $\lim_{x\to a} f(x) = f(a)$.

ANSWERS TO SECTION QUIZ

1. x ; y and z .

2. $\partial^2 f/\partial y^2$; compute f_y and differentiate f_y with respect to y .

3. (a) $\lim_{h\to 0} \{[g(x,y,z + h) - g(x,y,z)]/h\}$

(b) 0

4. $(y^2 e^{3x} - e^x)\sin(e^x y) - 3ye^{2x}\cos(e^x y)$

5. (a) Yes

(b) Only if the second partial with respect to x is continuous, and if the first partial with respect to y is continuous.

6. (a) $-(g + h)/d^2$

(b) Holding h and d constant, the barber will cut 7% more hair than requested for every minute he gossips with a customer.

(c) The barber will cut off all of the customer's hair.

15.2 Linear Approximations and Tangent Planes

PREREQUISITES

1. Recall how to compute a partial derivative (Section 15.1).
2. Recall how to find the linear approximation of a function of one variable (Section 1.6).
3. Recall how to find the tangent line to a curve (Section 1.6).

PREREQUISITE QUIZ

1. Use the linear approximation to estimate the value of $(1.02)^2$.
2. What is the equation of the tangent line to the curve described by $y = x^3 + x^2 + 1$ at $x = 2$?
3. Find $\partial f/\partial x$ and f_y for the following functions:

   (a) $f(x,y) = x + y \cos x$

   (b) $f(x,y) = y - 5$

GOALS

1. Be able to find the linear approximation for a function of two or three variables.
2. Be able to compute the tangent plane of the graph of functions of two or three variables.

STUDY HINTS

1. Linear approximation. With one variable calculus, we began at $f(x_0)$ and moved along the tangent line as we made a change in the x-direction to obtain an approximation for $f(x)$. Now, we need to make changes in other directions than just the x-direction to obtain an approximation for f. Thus, we get $f(x,y) \approx f(x_0,y_0) + (\partial f/\partial x)\Delta x +$

1. (continued)

$(\partial f/\partial y)\Delta y$ and $f(x,y,z) \approx f(x_0,y_0,z_0) + (\partial f/\partial x)\Delta x + (\partial f/\partial y)\Delta y + (\partial f/\partial z)\Delta z$, where the partials need to be evaluated at (x_0,y_0) and (x_0,y_0,z_0), respectively. As with one variable calculus, using a calculator defeats the purpose.

2. Tangent plane. The equation of a tangent plane is the same equation which is used to compute the linear approximation. If $f(x,y) = z$, the linear approximation equation $z = z_0 + f_x\Delta x + f_y\Delta y$ can be rearranged to read $0 = -f_x\Delta x - f_y\Delta y + \Delta z$. Thus, a normal vector to this plane is $-f_x\underline{i} - f_y\underline{j} + \underline{k}$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The tangent plane is given by $z = f(x_0,y_0) + [f_x(x_0,y_0)](x - x_0) + [f_y(x_0,y_0)](y - y_0)$. $\partial z/\partial x = 3x^2 - 6y$ and $\partial z/\partial y = 3y^2 - 6x$. At $(1,2,-3)$, $\partial z/\partial x = -9$ and $\partial z/\partial y = 6$. Thus, the tangent plane at $(1,2,-3)$ is $z = -3 + (-9)(x - 1) + 6(y - 2)$ or $z = -9x + 6y - 6$.

5. Using the tangent plane formula from Exercise 1, we have $f_x = 2x$ and $f_y = 6y^2$. This gives $z = 4 + 2(x - 1) + 6(y - 1) = 2x + 6y - 4$.

9. We could apply the method of Exercise 1 with $z_0 = f(x_0,y_0)$. However, note that the equation describes a plane, which is its own tangent plane.

13. A normal vector is $-f_x(x_0,y_0)\underline{i} - f_y(x_0,y_0)\underline{j} + \underline{k}$. To normalize it, divide by its length, which is $[(f_x(x_0,y_0))^2 + (f_y(x_0,y_0))^2 + 1]^{1/2}$. Using the result of Exercise 9, we get $\underline{n} = (-\underline{i} + \underline{j} + \underline{k})/\sqrt{1 + 1 + 1} = (-1/\sqrt{3})(\underline{i} - \underline{j} - \underline{k})$.

17. Use $f(x,y) \approx f(x_0,y_0) + [f_x(x_0,y_0)](x - x_0) + [f_y(x_0,y_0)](y - y_0)$. Let $f(x,y) = x^2(1 - \sqrt{y})$, so $f_x = 2x(1 - \sqrt{y})$ and $f_y = -x^2/(2\sqrt{y})$. Also, let $x_0 = 1$, $y_0 = 1.96 = (1.4)^2$, so $f(x,y) \approx f(1,1.96) +$

17. (continued)

$f_x(1,1.96)(x-1) + f_y(1,1.96)(y-1.96) = 1(1-1.4) + 2(1-1.4) \times (x-1) - 1/(2.8)(y-1.96) = -0.4 - 0.8(x-1) - (y-1.96)/(2.8)$. Then $(1.01)^2(1-\sqrt{1.98}) = f(1.01,1.98) \approx -0.4 - 0.8(0.01) - (0.02)/(2.8) \approx -0.408 - 0.007 \approx -0.415$.

21. Use $f(x,y,z) \approx f(x_0,y_0,z_0) + [f_x(x_0,y_0,z_0)](x-x_0) + [f_y(x_0,y_0,z_0)] \times (y-y_0) + [f_z(x_0,y_0,z_0)](z-z_0)$. Let $f(x,y,z) = xyz$, so $f_x = yz$, $f_y = xz$, and $f_z = xy$. Also, let $x_0 = 1 = y_0 = z_0$, so $f(x,y,z) \approx f(1,1,1) + [f_x(1,1,1)](x-1) + [f_y(1,1,1)](y-1) + [f_z(1,1,1)](z-1)$. Then $(0.98)(0.99)(1.03) = f(0.98,0.99,1.03) \approx 1 + (-0.02) + (-0.01) + (0.03) = 1.00$.

25. Since a cube has six faces, $f(a,v) = 6v/a$. Therefore, $f_a = -6v/a^2$ and $f_v = 6/a$. Also let $a_0 = 6$ and $v_0 = 1$. According to the linear approximation, $f(a,v) \approx f(6,1) + f_a(a-6) + f_v(v-1) = 1 - (a-6)/6 + (v-1)$. When $a = 6 + \Delta a$ and $v = 1 + \Delta v$, $f(6+\Delta a, 1+\Delta v) \approx 1 - \Delta a/6 + \Delta v$.

29. Let $\underline{u}$ denote the vector $(x_0,y_0,f(x_0,y_0))$. $f_x = x/(1-x^2-y^2)^{1/2}$ and $f_y = y/(1-x^2-y^2)^{1/2}$. A normal to the surface and the tangent plane at (x_0,y_0) is $(-x_0/(1-x_0^2-y_0^2)^{1/2}, -y_0/(1-x_0^2-y_0^2)^{1/2}, 1)$. Multiply this by $f(x_0,y_0)$ to produce $\underline{u}$. Hence these vectors are parallel, so $\underline{u}$ is also normal to the plane.

SECTION QUIZ

1. A box has dimensions $2.015 \times 0.998 \times 1.007$. Use the linear approximation to estimate its volume.

2. Estimate the surface area of the box in Question 1.

3. (a) Find the equation of the tangent plane to the surface $5x^2 + xy^3 - y - z = 0$ at $(1,1,5)$.

   (b) How does the tangent plane in part (a) compare with that to the surface $z = 5x^2 + xy^3 - y$ at $(1,1)$?

   (c) Find a normal unit vector to the plane in part (a).

4. What is the tangent plane to the unit sphere at $(1/2,-1/2,-1/\sqrt{2})$?

5. Hillbillies' feuding level (HFL) is known to be determined by $HFL = s^3t\sqrt{d}$, where s is the number of times family members have seen each others' faces during the day, t is the number of hours since the last attack, and d is the distance (in meters) between the homes.

   (a) Suppose $s = 4$, $t = 3.1$, and $d = 102$. Use the linear approximation to estimate HFL.

   (b) Hillbillies usually start throwing rocks at each other when $HFL > 1000$. If $s = 4$ and $t = 3.1$, approximate how close the homes can be without any rock throwing. (Hint: Guess a d_0 for the linear approximation formula and then find a Δd.)

ANSWERS TO PREREQUISITE QUIZ

1. 1.04

2. $y = 16x - 21$

3. (a) $\partial f/\partial x = 1 - y \sin x$; $f_y = \cos x$.

   (b) $\partial f/\partial x = 0$; $f_y = 1$.

ANSWERS TO SECTION QUIZ

1. 2.025

2. 10.090

3. (a) $11x + 2y - z = 8$

   (b) They are the same.

   (c) $(11\underline{i} + 2\underline{j} - \underline{k})/\sqrt{126}$

4. $x - y - \sqrt{2}z - 2 = 0$

5. (a) 1993.6

   (b) 25.0

15.3 The Chain Rule

PREREQUISITES

1. Recall how to use the chain rule (Section 2.2).
2. Recall the linear approximation formula for two variables (Section 15.2).

PREREQUISITE QUIZ

1. Write an expression for $(d/dx)f(g(x))$.
2. Differentiate $(\sqrt{x} + 1)^2/\exp(\cos x)$.
3. If $f(x,y) = x^2 + xy$, approximate $f(1.1,0.9)$ by using the linear approximation.

GOALS

1. Be able to use the chain rule to compute the partial derivatives of a function which depends on intermediate variables.
2. Be able to use the chain rule to find tangents to curves in graphs.

STUDY HINTS

1. Chain rule. This is a simple extension of the one-variable case. An extra term is added for each variable. Notice that each term has the same variable in the numerator and the denominator. For example, in $du/dt = (\partial u/\partial x)(dx/dt) + (\partial u/\partial y)(dy/dt) + (\partial u/\partial z)(dz/dt)$, there is an x in the numerator and the denominator of the first term. The rule is similar in structure to the chain rule for one variable. These suggestions should make the formula easy to memorize.

2. Tangent lines. If a line is tangent to a curve in a surface, the line must lie in the tangent plane. Convince yourself of this fact.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. (a) $\partial T/\partial x = 2x \cos y - y^2 \cos x$; $\partial T/\partial y = -x^2 \sin y - 2y \sin x$; $dx/dt = 8$; and $dy/dt = -2$. By the chain rule, $dT/dt = (\partial T/\partial x)(dx/dt) + (\partial T/\partial y)(dy/dt) = 16x \cos y - 8y^2 \cos x + 2x^2 \sin y + 4y \sin x =$ $(48 + 128t)\cos(3 - 2t) - 8(3 - 2t)^2 \cos(3 + 8t) +$ $2(3 + 8t)^2 \sin(3 - 2t) + (12 - 8t)\sin(3 + 8t)$.

(b) Substitution yields $T = (3 + 8t)^2 \cos(3 - 2t) - (3 - 2t)^2 \sin(3 + 8t)$. Therefore, $dT/dt = 2(8)(3 + 8t)\cos(3 - 2t) + 2(3 + 8t)^2 \sin(3 - 2t) + 2(2)(3 - 2t)\sin(3 + 8t) - (3 - 2t)^2 (8)\cos(3 + 8t)$, which is the same as in part (a).

5. We find $f'(t)$ in two ways. First we compute f_x , f_y , f_z , dx/dt , dy/dt , and dz/dt and combine them according to the chain rule. Secondly, we substitute $\underline{\sigma}(t)$ into the expression for f and differentiate. The two derivatives will be equal. $f_x = 1$; $f_y = 2y$; and $f_z = 3z^2$. Also, $dx/dt = -\sin t$; $dy/dt = \cos t$; and $dz/dt = 1$. Then the chain rule says $f'(t) = -\sin t + 2y \cos t + 3z^2$.

Substituting for x , y , and z gives $f(t) = \cos t + \sin^2 t + t^3$. Therefore $f'(t) = -\sin t + 2 \sin t \cos t + 3t^2$. Since $y = \sin t$ and $z = t$, $-\sin t + 2 \sin t \cos t + 3t^2 = -\sin t + 2y \cos t + 3z^2$.

9. Let $x = f(t)$ and $y = g(t)$. Then $z = f(t)/g(t) = x/y$. Applying the chain rule, we have $dz/dt = (\partial z/\partial x)\cdot(dx/dt) + (\partial z/\partial y)\cdot(dy/dt) = (1/y)\cdot f'(t) + (-x/y^2)g'(t) = [yf'(t) - xg'(t)]/y^2 = [g(t)f'(t) - f(t)g'(t)]/[g(t)]^2$, which is the quotient rule.

13. Recall that a normal vector to $z = f(x,y)$ is $-f_x(x_0,y_0)\underline{i} - f_y(x_0,y_0)\underline{j} + \underline{k}$. For $z = x^2 + y^2$, $f_x = 2x$ and $f_y = 2y$. Thus, a normal vector at $(1,2,5)$ is $-2\underline{i} - 4\underline{j} + \underline{k}$. All tangent vectors, $a\underline{i} + b\underline{j} + c\underline{k}$, lie in the tangent plane, which is $z = 5 + 2(x - 1) + 4(y - 2)$. The tangent vectors satisfy $-2a - 4b + c = 0$.

17. Let $x = f(t)$, $y = g(t)$, and $z = h(t)$. Then $u = xy/z$ can be differentiated by the chain rule: $du/dt = (\partial u/\partial x)(dx/dt) + (\partial u/\partial y)(dy/dt) + (\partial u/\partial z)(dz/dt) = (y/z)f'(t) + (x/z)g'(t) + (-xy/z^2)h'(t)$. Putting this over a common denominator, z^2 , yields $du/dt = [yzf'(t) + xzg'(t) - xyh'(t)]/z^2 = [f'(t)g(t)h(t) + f(t)g'(t)h(t) - f(t)g(t)h'(t)]/[h(t)]^2$.

21. The half-line is parametrized by (sx_0,sy_0,sz_0) , $s \geqslant 0$. This curve passes through (x_0,y_0,z_0) at $s = 1$ and by assumption, the entire half-line lies in the surface $z = f(x,y)$. By the box on p. 782, the tangent to this curve lies in the tangent plane. But the tangent at $s = 1$ is the vector (x_0,y_0,z_0) . Thus, (x_0,y_0,z_0) lies in the tangent plane. Parametrizing the half-line by $(s\alpha x_0,s\alpha y_0,s\alpha z_0)$ for fixed $\alpha > 0$ and variable $s \geqslant 0$, we similarly see that $(\alpha x_0,\alpha y_0,\alpha z_0)$ lies in the tangent plane. Thus the whole half-line lies in it.

An example of such a surface is $z = \sqrt{x^2 + y^2}$.

SECTION QUIZ

1. (a) Suppose that P is a function of four variables, w , x , y , and z . If w , x , y , and z are all functions of r , what do you suppose is the chain rule formula for dP/dr ?

1. (b) Let $P = wz + yx$, $w = 3r$, $x = \cos r$, $y = r^4$, and $z = 1/r$. Find dP/dr by using the chain rule formula from part (a).

   (c) Use substitution to check your answer in part (b).

2. Suppose that $u = f(x,y)$, $x = g(t)$, and $y = h(t)$. Write a formula for d^2u/dt^2 . [Hint: Your formula should involve $d(\partial u/\partial x)/dt$.]

3. The curve described parametrically by $(x,y,z) = (2 \cos t, 3 \sin t, t)$ lies on the elliptical cylinder $9x^2 + 4y^2 = 36$. Show that the tangent line to the curve lies on the tangent plane to the surface at the point $(\sqrt{2}, 3\sqrt{2}/2, \pi/4)$.

4. The flight speed of Santa's reindeer varies according to w , the weight of the toys they are hauling, and D , the distance they have travelled on Christmas eve. In addition, it is also known that the weight of the toys decreases with time and the distance travelled is a function of time.

   (a) Let t represent the time elapsed since the reindeer left the North Pole and let s be their speed at time t . Using the chain rule, write a formula for ds/dt .

   (b) Compute ds/dt in terms of the partials of s if $w = -50t$ and $D = 75t$.

ANSWERS TO PREREQUISITE QUIZ

1. $f'(g(x)) \cdot g'(x)$

2. $[1 + 1/\sqrt{x} + (\sin x)(\sqrt{x} + 1)^2]/\exp(\cos x)$

3. 2.2

ANSWERS TO SECTION QUIZ

1. (a) $dP/dr = (\partial P/\partial w)(dw/dr) + (\partial P/\partial x)(dx/dr) + (\partial P/\partial y)(dy/dr) + (\partial P/\partial z)(dz/dr)$

   (b) $4r^3\cos r - r^4\sin r$

   (c) $4r^3\cos r - r^4\sin r$

2. $[d(\partial u/\partial x)/dt](dx/dt) + (\partial u/\partial x)(d^2x/dt^2) + [d(\partial u/\partial y)/dt](dy/dt) + (\partial u/\partial y)(d^2y/dt^2)$

3. Tangent plane is $18\sqrt{2}x + 12\sqrt{2}y - z = 72 - \pi/4$; tangent line is $(-\sqrt{2},3\sqrt{2}/2,1)(t - \pi/4) + (\sqrt{2},3\sqrt{2}/2,\pi/4)$. Tangent line satisfies tangent plane equation.

4. (a) $ds/dt = (\partial s/\partial w)(dw/dt) + (\partial s/\partial D)(dD/dt)$

   (b) $ds/dt = (\partial s/\partial w)(-50) + (\partial s/\partial D)(75)$

15.4 Matrix Multiplication and the Chain Rule

PREREQUISITES

1. Recall how to find a partial derivative (Section 15.1).
2. Recall how to use the chain rule (Section 15.3).

PREREQUISITE QUIZ

1. Find $\partial f/\partial x$, $\partial f/\partial y$, and $\partial^2 f/\partial x^2$ for the following functions:
   (a) $f(x,y) = x^2 + 3x^2y - y^2 + y$
   (b) $f(x,y) = e^{x^2y} - \sin 3y^2$
2. Suppose that f is a function of x and y, and that x and y are both functions of t. Write a formula for df/dt.
3. Let $f(x,y) = x^3 + y^2$, $x = e^t$, and $y = t^2$.
   (a) Substitute for x and y, and then compute df/dt.
   (b) Use the chain rule to compute df/dt.

GOALS

1. Be able to multiply matrices.
2. Be able to write down the derivative matrix and use it to obtain the general chain rule.

STUDY HINTS

1. Matrices. A matrix is a rectangular array of numbers. An $n \times m$ matrix has n rows and m columns. You should remember that we talk about rows before columns. Thus $n \times m$ means n rows and m columns. The (i,j) entry is the number located in row i, column j.
2. Matrix multiplication. To obtain the product of matrix A and matrix B, we manipulate the entries of row i in A and column j in B to

2. (continued)

obtain the ij entry of AB. Therefore, the number of columns of A must equal the number of rows of B if AB is to exist. If A is an $n \times m$ matrix and B is $m \times p$, then AB is an $n \times p$ matrix (The m's "cancel").

3. Notation. $\partial u/\partial(x,y,z)$ is the vector consisting of the partials of u with respect to x, y, and z. $\partial(x,y,z)/\partial t$ is the vector consisting of the partials of x, y, and z with respect to t. In general, similar notation defines a matrix where all of the top variables are differentiated with respect to all of the bottom variables.

4. Derivative matrix. In the notation $\partial(u_1, \ldots, u_m)/\partial(x_1, \ldots, x_n)$, the matrix is written so that the u's are listed down a column. The x's are listed across a row. Another way to help you remember is to realize that an $m \times n$ matrix is required. When you evaluate the matrix at a given point, the point refers to the x's, not the u's.

5. Non-commutativity. Example 5 shows that $AB \neq BA$ in general. Matrix multiplication is not commutative.

6. General chain rule. The formula for the general chain rule is $\partial(u_1, \ldots, u_m)/\partial(t_1, \ldots, t_k) = [\partial(u_1, \ldots, u_m)/\partial(x_1, \ldots, x_n)] \times [\partial(x_1, \ldots, x_n)/\partial(t_1, \ldots, t_k)]$, where "$\times$" indicates matrix multiplication. Note how cancellations can be made in this formula as if it was a "fraction." The general formula is the product of two derivative matrices, so any desired partial may be obtained by multiplication.

7. Second derivatives by chain rule. There are some subtleties which you should be aware of when you compute second derivatives. Suppose that $z = f(x,y)$ and $x = r\cos\theta$, $y = r\sin\theta$. Then $\partial z/\partial\theta =$

7. (continued)

$(\partial z/\partial x)(\partial x/\partial\theta) + (\partial z/\partial y)(\partial y/\partial\theta)$. Now, $\partial^2 z/\partial\theta^2$ is obtained by using the product and sum rules to get $\partial^2 z/\partial\theta^2 = (\partial^2 z/\partial x\partial\theta)(\partial x/\partial\theta) + (\partial z/\partial x)(\partial^2 x/\partial\theta^2) + (\partial^2 z/\partial y\partial\theta)(\partial y/\partial\theta) + (\partial z/\partial y)(\partial^2 y/\partial\theta^2)$. One of the terms contains $\partial^2 z/\partial x\partial\theta$, and another contains $\partial^2 z/\partial y\partial\theta$, which both involve the expression which we obtained for $\partial z/\partial\theta$. In the second step, many students forget that z depends on θ .

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $[1\ 2\ 3]\begin{bmatrix}4\\5\\6\end{bmatrix} = [1\cdot 4 + 2\cdot 5 + 3\cdot 6] = [32]$.

5. Use the method of Example 2.
$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{bmatrix} = \begin{bmatrix}\sin v & u\cos v\\ v\exp(uv) & u\exp(uv)\end{bmatrix}$, which is $\begin{bmatrix}\sin 1 & 0\\ 1 & 0\end{bmatrix}$ at $(0,1)$.

9. $\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix} = \begin{bmatrix}1\cdot 2 + 2\cdot 4 & 1\cdot 3 + 2\cdot 5\\ 0\cdot 2 + 1\cdot 4 & 0\cdot 3 + 1\cdot 5\end{bmatrix} = \begin{bmatrix}10 & 13\\ 4 & 5\end{bmatrix}$.

13. $\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$ is undefined because the number of columns in the first matrix (two) is unequal to the number of rows of the second matrix (three).

17. $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}1\cdot a + 0\cdot c & 1\cdot b + 0\cdot d\\ 0\cdot a + 0\cdot c & 0\cdot b + 0\cdot d\end{bmatrix} = \begin{bmatrix}a & b\\ 0 & 0\end{bmatrix}$.

21. Use the formula $[\partial z/\partial x\ \ \partial z/\partial y] = [\partial z/\partial u\ \ \partial z/\partial v]\begin{bmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{bmatrix}$. By matrix multiplication, $[\partial z/\partial x\ \ \partial z/\partial y] = [2u\ \ 2v]\begin{bmatrix}2 & 0\\ 3 & 1\end{bmatrix} = [4u + 6v\ \ \ 2v]$. In terms of x and y , we have $\partial z/\partial x = 26x + 6y + 70$ and $\partial z/\partial y =$

21. (continued)

$6x + 2y + 14$.

Using direct substitution, $z = (2x + y)^2 + (3x + y + 7)^2$. Thus, $\partial z/\partial x = 2(2x + 7)(2) + 2(3x + y + 7)(3) = 26x + 6y + 70$, and $\partial z/\partial y = 2(3x + y + 7) = 6x + 2y + 14$. The two answers are consistent.

25. (a) $\partial(x,y)/\partial(t,s) = \begin{bmatrix} \partial x/\partial t & \partial x/\partial s \\ \partial y/\partial t & \partial y/\partial x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. $\partial(u,v)/\partial(x,y) =$

$\begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial u/\partial y \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$.

(b) $u = (t + s)^2 + (t - s)^2$; $v = (t + s)^2 - (t - s)^2$.

$\partial(u,v)/\partial(t,s) = \begin{bmatrix} \partial u/\partial t & \partial u/\partial s \\ \partial v/\partial t & \partial v/\partial s \end{bmatrix} =$

$\begin{bmatrix} 2(t + s) + 2(t - s) & 2(t + s) - 2(t - s) \\ 2(t + s) - 2(t - s) & 2(t + s) + 2(t - s) \end{bmatrix} = \begin{bmatrix} 4t & 4s \\ 4s & 4t \end{bmatrix}$.

(c) We wish to show that $\partial(u,v)/\partial(t,s) = [\partial(u,v)/\partial(x,y)][\partial(x,y)/\partial(t,s)]$.

The right side is $\begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2x + 2y & 2x - 2y \\ 2x - 2y & 2x + 2y \end{bmatrix}$.

But $2x + 2y = 4t$ and $2x - 2y = 4s$, so the above $= \begin{bmatrix} 4t & 4s \\ 4s & 4t \end{bmatrix} =$

$\partial(u,v)/\partial(t,s)$.

29. By the general chain rule, $[\partial u/\partial r \ \partial u/\partial \theta \ \partial u/\partial \phi] = [\partial u/\partial x \ \partial u/\partial y \ \partial u/\partial z] \times$

$\begin{bmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial \phi \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial \phi \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial \phi \end{bmatrix} = [\partial u/\partial x \ \partial u/\partial y \ \partial u/\partial z] \times$

$\begin{bmatrix} \cos\theta \sin\phi & -r \sin\theta \sin\phi & r \cos\theta \cos\phi \\ \sin\theta \sin\phi & r \cos\theta \sin\phi & r \sin\theta \cos\phi \\ \cos\phi & 0 & -r \sin\phi \end{bmatrix}$. Thus, $\partial u/\partial r =$

29. (continued)

$\cos\theta \sin\phi(\partial u/\partial x) + \sin\theta \sin\phi\, (\partial u/\partial y) + \cos\phi(\partial u/\partial z)$; $\partial u/\partial\theta =$ $-r\sin\theta\sin\phi(\partial u/\partial x) + r\cos\theta\sin\phi(\partial u/\partial y)$; and $\partial u/\partial\phi =$ $r\cos\theta\cos\phi(\partial u/\partial x) + r\sin\theta\cos\phi(\partial u/\partial y) - r\sin\phi(\partial u/\partial z)$.

33. We have $b_i = 1/m$ for $i = 1, \ldots, m$, so $AB = \Sigma_{i=1}^{m} a_i(1/m) = (1/m)\Sigma_{i=1}^{m} a_i$. It is just the average of the entries of A .

37. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; that is, $\begin{bmatrix} a+2b & 2a+5b \\ c+2d & 2c+5d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence $a + 2b = 1$(i) ; $2a + 5b = 0$(ii) ; $c + 2d = 0$(iii) ; and $2c + 5d = 1$(iv) . Subtracting 2 times equation (i) from equation (ii), we get $b = -2$, so $a = 5$. Subtracting 2 times equation (iii) from equation (iv), we get $d = 1$; so $c = -2$. Hence $B = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$.

41. (a) $\left|\partial(x,y)/\partial(t,s)\big|_{(1,2)}\right| = \left|\begin{bmatrix} 2t & 2s \\ 2t & -2s \end{bmatrix}\Bigg|_{(1,2)}\right| = \begin{vmatrix} 2 & 4 \\ 2 & -4 \end{vmatrix} = -16$.

(b) $\left|\partial(u,v)/\partial(x,y)\big|_{(5,-3)}\right| = \left|\begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}\Bigg|_{(5,-3)}\right| = (x - y)\big|_{(5,-3)} = 8$.

(c) We have $u = 2t^2$ and $v = t^4 - s^4$, so $\left|\partial(u,v)/\partial(t,s)\big|_{(1,2)}\right| = \left|\begin{bmatrix} 4t & 0 \\ 4t^3 & -4s^3 \end{bmatrix}\Bigg|_{(1,2)}\right| = \begin{vmatrix} 4 & 0 \\ 4 & -32 \end{vmatrix} = 128$, which equals $-16\cdot 8$.

45. (a) $x/au = \sin\phi\cos\theta$; $y/bu = \sin\phi\sin\theta$; and $z/cu = \cos\phi$. Compute $(x/au)^2 + (y/bu)^2 + (z/cu)^2 = \sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi = \sin^2\phi(\cos^2\theta + \sin^2\phi) + \cos^2\phi = \sin^2\phi + \cos^2\phi = 1$. For

45. (a) (continued)

each constant value of u, this describes an ellipsoid centered at the origin.

(b) $x/(au \sin\phi) = \cos\theta$; and $y/(bu \sin\phi) = \sin\theta$. Therefore, $(x/au \sin\phi)^2 + (y/bu \sin\phi)^2 = 1$. But $z/(cu \cos\phi) = 1$, so $(x/(au \sin\phi))^2 + (y/(bu \sin\phi))^2 = (z/(cu \cos\phi))^2$. Therefore, $(x/(a \sin\phi))^2 + (y/(b \sin\phi))^2 = (z/(c \cos\phi))^2$. If ϕ is constant, then for each value of ϕ, the above equation describes an elliptical cone with vertex at the origin.

(c) Substitute $u \sin\phi = x/(a \cos\theta)$ into $y = xb \sin\theta/a \cos\theta$. If θ is constant, this describes a line through the origin in the xy-plane, or a plane in three-dimensional space.

(d) $|\partial(x,y,z)/\partial(u,\phi,\theta)| =$

$$\begin{vmatrix} a\sin\phi\cos\theta & au\cos\phi\cos\theta & -au\sin\phi\sin\theta \\ b\sin\phi\sin\theta & bu\cos\phi\sin\theta & bu\sin\phi\cos\theta \\ c\cos\phi & -cu\sin\phi & 0 \end{vmatrix} =$$

$a \sin\phi \cos\theta(bcu^2 \sin^2\phi \cos\theta) +$

$au \cos\phi \cos\theta(bcu \sin\phi \cos\phi \cos\theta) -$

$au \sin\phi \sin\theta(-bcu \sin^2\phi \sin\theta - bcu \cos^2\phi \sin\theta) =$

$abcu^2 \sin\phi \cos^2\theta(\sin^2\phi + \cos^2\phi) + abcu^2 \sin\phi \sin^2\theta(\sin^2\phi + \cos^2\phi) =$

$abcu^2 \sin\phi(\cos^2\theta + \sin^2\theta) = abcu^2 \sin\phi$.

49. Simpson's rule says that $\int_a^b f(x)dx \approx [(b-a)/(3n)][f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$. In matrix form, this is

$$\int_a^b f(x)dx \approx [(b-a)/(3n)][f(x_0)\ f(x_1)\ \dots\ f(x_n)] \begin{bmatrix} 1 \\ 4 \\ 2 \\ \vdots \\ 2 \\ 4 \\ 1 \end{bmatrix}.$$

SECTION QUIZ

1. True or false.

(a) An $n \times m$ matrix has n columns and m rows.

(b) An $n \times m$ matrix has n rows and m columns.

(c) The (i,j) entry is located in row i, column j.

(d) The (i,j) entry is located in column i, row j.

2. The following matrix sizes for A and B are given. Does AB exist? If yes, what is the size of AB?

(a) A: $n \times m$; B: $m \times n$

(b) A: $n \times m$; B: $n \times m$

(c) A: $n \times m$; B: $m \times p$

3. Write the matrix $\partial(u,v,w)/\partial(x,y)$ for $u(x,y) = x^2 + y^2$, $v(x,y) = -3x^2y$, and $w(x,y) = y^2$.

4. Game show contestants are chosen according to how enthusiastic they are, E, and their willingness to perform abnormal acts, P. Now, E and P are both functions of greed, g, and insanity level, i. Finally, both g and i are functions of wealth, $\$$, the contestant's age, a, and the number of children under twenty years of age which drive them crazy, c.

(a) What is $\partial(E,P)/\partial(g,i)$ in matrix form?

(b) What is $\partial(g,i)/\partial(\$,a,c)$ in matrix form?

(c) How is $\partial(E,P)/\partial(\$,a,c)$ related to your answers in parts (a) and (b)?

ANSWERS TO PREREQUISITE QUIZ

1. (a) $\partial f/\partial x = 2x + 6xy$; $\partial f/\partial y = 3x^2 - 2y + 1$; $\partial^2 f/\partial x^2 = 2 + 6y$

(b) $\partial f/\partial x = 2xy \exp(x^2y)$; $\partial f/\partial y = x^2\exp(x^2y) - 6y \cos 3y^2$; $\partial^2 f/\partial x^2 = (2y + 2xy)\exp(x^2y)$

2. $df/dt = (\partial f/\partial x)(dx/dt) + (\partial f/\partial y)(dy/dt)$

3. (a) $f(x,y) = e^{3t} + t^4$, so $df/dt = 3e^{3t} + 4t^3$.

   (b) $df/dt = (\partial f/\partial x)(dx/dt) + (\partial f/\partial y)(dy/dt) = (3x^2)(e^t) + (2y)(2t) = 3e^{3t} + 4t^3$

ANSWERS TO SECTION QUIZ

1. (a) False

   (b) True

   (c) True

   (d) False

2. (a) Yes; $n \times n$.

   (b) No

   (c) Yes; $n \times p$.

3. $$\begin{bmatrix} 2x & 2y \\ -6xy & -3x^2 \\ 0 & 2y \end{bmatrix}$$

4. (a) $$\begin{bmatrix} \partial E/\partial g & \partial E/\partial i \\ \partial P/\partial g & \partial P/\partial i \end{bmatrix}$$

   (b) $$\begin{bmatrix} \partial g/\partial \$ & \partial g/\partial a & \partial g/\partial c \\ \partial i/\partial \$ & \partial i/\partial a & \partial i/\partial c \end{bmatrix}$$

   (c) Answer in (a) multiplied by answer in (b).

15.R Review Exercises for Chapter 15

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Holding y constant, we get $g_x = \pi \cos(\pi x)/(1 + y^2)$. And holding x constant, we get $g_y = -2y \sin(\pi x)/(1 + y^2)^2$.

5. Holding y and z constant and differentiating gives $h_x = z$. Holding x and z constant, we get $h_y = 2y + z$. Holding x and y constant, we get $h_z = x + y$.

9. Recall that $(d/dx)\int_0^x f(t)dt = f(x)$, so $g_x = z + e^z x^2 e^x = z + x^2 e^{x+z}$; $g_y = 0$; and $g_z = x + e^z \int_0^x t^2 e^t dt$.

13. $\partial u/\partial x = z^2 + z^3 \sin(xz^3)$, so $\partial^2 u/\partial x \partial z = 2z + 3z^2 \sin(xz^3) + 3xz^5 \cos(xz^3)$. On the other hand, $\partial u/\partial z = 2xz + 3xz^2 \sin(xz^3)$, so $\partial^2 u/\partial z \partial x = 2z + 3z^2 \sin(xz^3) + 3xz^5 \cos(xz^3)$, which is also $\partial^2 u/\partial x \partial z$.

17. $(\partial/\partial x)\exp(x - \cos(yx)) = (1 + y \sin(yx))\exp(x - \cos(yx))$. Evaluating at $(1,0)$, we get $\exp(1 - \cos(0)) = e^0 = 1$.

21. (a) Substitute $x = 27$ and $V = 65$ into $T = 32V/(x + 32)$ to get $(32)(65)/(27 + 32) = 2080/59 \approx 35.25$ minutes.

(b) (i) $\partial T/\partial x = -32V/(x + 32)^2$. Substitute $x = 27$ and $V = 65$, giving $(\partial T/\partial x)|_{(27,65)} = -32 \cdot 65/(27 + 32)^2 = -2080/(59)^2 \approx -0.598$ minutes/foot. This means that when you are diving at a depth of 27 feet, going a foot deeper decreases the possible time of the dive by 0.598 minutes, about 36 seconds.

(ii) $\partial T/\partial V = 32/(x + 32)$. Substitute $x = 27$, giving $(\partial T/\partial V)|_{x=27} = 32/59 \approx 0.542$ minutes/cubic foot. This means that when you are diving at a depth of 27 feet, bringing an extra cubic foot of air along extends the possible time of the dive by 0.542 minutes, about 33 seconds.

25. For the limit to exist, we must get the same value no matter how we approach $(0,0)$. Suppose we approach $(0,0)$ along the line $y = rx$, where r is any real number. Then $\lim_{(x,y)\to(0,0)} [(x^3 - y^3)/(x^2 + y^2)] = \lim_{(x,rx)\to(0,0)} [(x^3 - r^3x^3)/(x^2 + r^2x^2)] = [(1 - r^3)/(1 + r^2)] \lim_{x\to 0} x = 0$.

29. The tangent plane is given by $z = f(x_0,y_0) + [f_x(x_0,y_0)](x - x_0) + [f_y(x_0,y_0)](y - y_0)$. Here, $f_x = ye^{xy}$ and $f_y = xe^{xy}$, so at $(0,0)$, $f_x = f_y = 0$ and $f = 1$. Thus, the tangent plane is $z = 1$.

33. The linear approximation for two variables is $\ell(x,y) = f(x_0,y_0) + [f_x(x_0,y_0)](x - x_0) + [f_y(x_0,y_0)](y - y_0)$. Let $f(x,y) = x^y$, so $f_x = yx^{y-1}$ and $f_y = (\ln x)x^y$. Let $(x,y) = (0.999,1.001)$ and $(x_0,y_0) = (1,1)$. Then the linear approximation is $1 + 1(-0.001) + 0 = 0.999$.

37. Rewriting $x^2 + 2y^2 + 3z^2 = 6$, we get $z = \sqrt{-x^2/3 - 2y^2/3 + 2}$. Thus, $f_x = -x/3z$ and $f_y = -2y/3z$. The normal is $-f_x\underline{i} - f_y\underline{j} + \underline{k}$. At $(1,1,1)$, the normal is $(1/3)\underline{i} + (2/3)\underline{j} + \underline{k}$. The particle travels along the line $x = 1 + t$, $y = 1 + 2t$, $z = 1 + 3t$, which intersects the surface $x^2 + y^2 + z^2 = 103$ at the point P when $(1 + t)^2 + (1 + 2t)^2 + (1 + 3t)^2 = 103$. Solve this: $3 + 12t + 14t^2 = 103$, so $7t^2 + 6t - 100 = 0$, giving $t = (-6 \pm \sqrt{36 + 2800})/14 = (-3 \pm 2\sqrt{709})/7$. Since $t > 0$, we take $t = (-3 + 2\sqrt{709})/7$. The distance from $(1,1,1)$ to P is $\sqrt{(x - 1)^2 + (y - 1)^2 + (z - 1)^2} = \sqrt{t^2 + 4t^2 + 9t^2} = t\sqrt{14} = \sqrt{14}(-3 + 2\sqrt{709})/7$. Hence, the time is $\sqrt{14}(-3 + 2\sqrt{709})/70$.

41. A function is constant if the derivative is 0. Differentiate $u(f(t),t) = u(x,t)$ by the chain rule to get $(\partial u/\partial x)(dx/dt) + (\partial u/\partial t)(dt/dt) = u_x(dx/dt) + u_t$. Since $dx/dt = u$, we get $uu_x + u_t$, which is given to be 0; therefore, $u(f(t),t)$ is constant in t.

45. Let $x = f(t)$, $y = g(t)$, and $u = \exp(xy)$. Then $du/dt = (\partial u/\partial x)(dx/dt) + (\partial u/\partial y)(dy/dt) = (y\exp(xy))(f'(t)) + (x\exp(xy)) \times (g'(t)) = [f'(t)g(t) + f(t)g'(t)]\exp[f(t)g(t)]$.

49. The tangent plane is given by $z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$. $f_x = 2x$ and $f_y = 12y$, so $z = 7 + 2(x-1) + 12(y-1) = 2x + 12y - 7$. In the xy-plane, $z = 0$, so the line is $2x + 12y = 7$, i.e., $y = -x/6 + 7/12$.

53. $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 - 1\cdot 0 & 0\cdot 0 + (-1)(-1) \\ 1\cdot 1 + 0\cdot 0 & 1\cdot 0 + 0(-1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

57. $$\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1(2)+2(-1) & 1(1)+2(-1) & 1(2)+2(1) \\ -1(2)+1(-1) & -1(1)+1(-1) & -1(2)+1(1) \\ 2(2)+1(-1) & 2(1)+1(-1) & 2(2)+1(1) \end{bmatrix} =$$
$$\begin{bmatrix} 0 & -1 & 4 \\ -3 & -2 & -1 \\ 3 & 1 & 5 \end{bmatrix}.$$

61. (a) $z = (u^2+v^2)/(u^2-v^2) = (e^{-2x-2y} + e^{2xy})/(e^{-2x-2y} - e^{2xy})$, so $\partial z/\partial x = [(-2e^{-2x-2y} + 2ye^{2xy})(e^{-2x-2y} - e^{2xy}) - (e^{-2x-2y} + e^{2xy}) \times (-2e^{-2x-2y} - 2ye^{2xy})]/(e^{-2x-2y} - e^{2xy})^2 = 4(e^{-2x-2y+2xy})(1+y)/(e^{-2x-2y} - e^{2xy})^2$. $\partial z/\partial y = 4(e^{-2y-2x+2xy})(1+x)/(e^{-2x-2y} - e^{2xy})^2$ because of the symmetry of x and y in u and v.

(b) By the chain rule, $\partial z/\partial x = (\partial z/\partial u)(\partial u/\partial x) + (\partial z/\partial v)(\partial v/\partial x) = [-4uv^2/(u^2-v^2)^2](-e^{-x-y}) + [4u^2v/(u^2-v^2)^2](ye^{xy}) = (4u^2v^2 + 4yu^2v^2)/(u^2-v^2)^2 = 4(e^{-2x-2y+2xy})(1+y)/(e^{-2x-2y} - e^{2xy})^2$. Similarly, $\partial z/\partial y = (\partial z/\partial u)(\partial u/\partial y) + (\partial z/\partial v)(\partial v/\partial y)$. Again, the symmetry of x and y in u and v gives $\partial z/\partial y = 4(e^{-2x-2y+2xy}) \times (1+x)/(e^{-2x-2y} - e^{2xy})^2$.

65. (a) Merely divide both sides of $PV = nRT$ by the appropriate factor to get: $n = PV/RT$; $P = nRT/V$; $T = PV/nR$; $V = nRT/P$.

65. (b) $\partial P/\partial T$ is the increase in pressure relative to the increase in (absolute) temperature, when the number of moles and the volume are held constant. Differentiate $P = nRT/V$ with respect to T: $\partial P/\partial T = nR/V$. For example, given two moles of a gas at $300^\circ K$, a constant volume of 4 liters, and a pressure of $150R$ Newtons/meter^2, the rate of change of pressure with temperature would be $2R/4 = R/2$. That is, a 1° change in temperature would produce a change of about $R/2$ (N/m^2) in pressure.

(c) Differentiate the appropriate equation from part (a): $\partial V/\partial T = nR/P$; $\partial T/\partial P = V/nR$; $\partial P/\partial V = -nRT/V^2$. Multiply to get $\partial V/\partial T \cdot \partial T/\partial P \cdot \partial P/\partial V = nR/P \cdot V/nR \cdot (-nRT)/V^2 = -nRT/PV = -1$, since $PV = nRT$.

69. Here, we are differentiating with respect to x to find $\partial w/\partial x$. At the same time, we hold y constant. However, $y = x^2$, and if y is constant, x can not be variable.

73. (a) $f_x = 2x/(x^2 + y^2)$, so $f_{xx} = [2(x^2 + y^2) - 2x(2x)]/(x^2 + y^2)^2 = (2y^2 - 2x^2)/(x^2 + y^2)^2$. Similarly, the symmetry of the function gives $f_{yy} = (2x^2 - 2y^2)/(x^2 + y^2)^2$. Therefore, $f_{xx} + f_{yy} = 0$.

(b) $g_x = (-1/2)(2x)/(x^2 + y^2 + z^2)^{3/2} = -x/(x^2 + y^2 + z^2)^{3/2}$, so $g_{xx} = [-(x^2 + y^2 + z^2)^{3/2} + x(3/2)(x^2 + y^2 + z^2)^{1/2}(2x)]/(x^2 + y^2 + z^2)^3 = (2x^2 - y^2 - z^2)/(x^2 + y^2 + z^2)^{5/2}$. By symmetry, $g_{yy} = (2y^2 - x^2 - z^2)/(x^2 + y^2 + z^2)^{5/2}$ and $g_{zz} = (2z^2 - x^2 - y^2)/(x^2 + y^2 + z^2)^{5/2}$. Therefore, $g_{xx} + g_{yy} + g_{zz} = 0$.

(c) $h_x = -2x/(x^2 + y^2 + z^2 + w^2)^2$, so $h_{xx} = [-2(x^2 + y^2 + z^2 + w^2)^2 + 2x(2)(x^2 + y^2 + z^2 + w^2)(2x)]/(x^2 + y^2 + z^2 + w^2)^4 = (6x^2 - 2y^2 - 2z^2 - 2w^2)/(x^2 + y^2 + z^2 + w^2)^3$. Similarly, symmetry yields $h_{yy} = (6y^2 - 2x^2 - 2z^2 - 2w^2)/(x^2 + y^2 + z^2 + w^2)^3$, $h_{zz} = (6z^2 - 2x^2 -$

73. (c) (continued)

$2y^2 - 2w^2)/(x^2 + y^2 + z^2 + w^2)^3$, and $h_{ww} = (6w^2 - 2x^2 - 2y^2 - 2z^2)/(x^2 + y^2 + z^2 + w^2)^3$. Therefore, $h_{xx} + h_{yy} + h_{zz} + h_{ww} = 0$.

77. $\partial^2 z/\partial x \partial y$ measures the rate of change of $\partial z/\partial y$, the slope of the graph in the y-direction, as x varies. From Fig. 15.R.1, we see that $\partial z/\partial y$ is increasing as x is increased, moving along the line $y = 0$. Thus, it is plausible that $(\partial^2 z/\partial x \partial y)|_{(0,0)} > 0$. In fact, we calculate $(\partial z/\partial y)|_{y=0} = x$ and so $(\partial^2 z/\partial x \partial y)|_{(0,0)} = 1$. Also from the graph, $\partial z/\partial x$ is decreasing as y is increased, moving along the line $x = 0$. Thus it is plausible that $(\partial^2 z/\partial y \partial x)|_{(0,0)} < 0$. In fact, $(\partial^2 z/\partial y \partial x)|_{(0,0)} = -1$ as above. Thus, the inequality of the mixed partials <u>is</u> consistent with the graph.

TEST FOR CHAPTER 15

1. True or false.

   (a) $\partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x$ if and only if $\partial f/\partial x = \partial f/\partial y$ at (x_0, y_0) .

   (b) If a curve lies on a surface, the tangent line to the curve at (x_0, y_0, z_0) always lies on the tangent plane to the surface at (x_0, y_0, z_0) .

   (c) $\partial(x,y,z)/\partial(r,s)$ can be represented by a matrix composed of 3 rows and 2 columns.

   (d) If $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, then $\lim_{x\to 0} f(x,0) = 0$.

   (e) If $\lim_{x\to 0} f(x,0) = 0$ and $\lim_{y\to 0} f(0,y) = 0$, then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

2. (a) Sketch the graph of $z = |y|$ in space.

   (b) Does $\partial z/\partial x$ exist at the origin? If yes, what is it?

   (c) Does $\partial z/\partial y$ exist at the origin? If yes, what is it?

3. (a) Compute $\partial^3 f/\partial x^2 \partial y$ and $\partial^3 f/\partial x \partial y^2$ for $f(x,y) = 5x^3 + 3y^2$.

   (b) Does the equality of mixed partials apply to the question in part (a)? Why or why not?

4. Recall that in spherical coordinates, $y = \rho \sin\phi \sin\theta$. Suppose that ρ , ϕ , and θ are functions of time, t . Find a formula for d^2y/dt^2 in terms of the first and second derivatives of ρ , ϕ , and θ .

5. (a) If $u = f(r,s,p)$, write $\partial u/\partial s$ as a limit.

   (b) For a function, $f(x,y)$, define what it means for f to be continuous at (x_0,y_0) .

6. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$. Compute AB and BA if they exist. If either does not exist, explain why.

7. Let $k = rm^2/p$.

   (a) Use the linear approximation to estimate k when $r = 2.1$, $m = 1.95$, and $p = 5.2$.

   (b) What is the tangent plane to k at $(r,m,p) = (5,1,2)$?

8. Let $z = f(x,y) = (x^2 - y^2)/(x^2 + y^2)$.

   (a) Find the vector which is normal to the surface at $(1,1,0)$.

   (b) Prove that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

9. (a) Compute $\partial t/\partial r$ if $r = st$.

   (b) Compute $D_3 f$ if $f(x,y,z) = \ln\sqrt{xyz}$.

   (c) Compute h_b if $h(b,k,g) = bk\sin(gb)$.

10. Teenagers are known to break out with zits according to the following equation: $z = (f^3 + cf)/w$, where z is the number of new pimples breaking out each month, f is the number of pounds of fatty foods eaten in the previous month, c is the number of ounces of chocolate eaten in the previous month, and w is the number of hours spent washing the face in the previous month.

(a) Find $\partial z/\partial f$ when $f = 10$, $c = 50$, and $w = 3$.

(b) What is the physical interpretation of $\partial z/\partial f$?

(c) Suppose f , c , and w are functions of h , the state of hunger. (w is a function of h because feeding oneself consumes washing time.) Write an expression for dz/dh .

(d) What would happen to a teenager's face if $w = 0$?

ANSWERS TO CHAPTER TEST

1. (a) False; only needs continuous second partial derivatives.

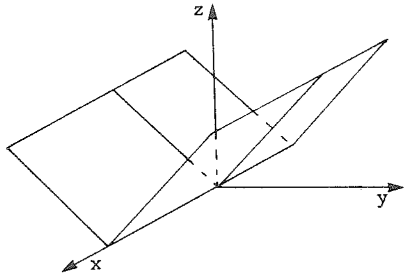
(b) True

(c) False; matrix is 3 columns by 2 rows.

(d) True

(e) False; let $f(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ 1 & \text{if } x \neq 0 \text{ or } y \neq 0 \end{cases}$.

2. (a)



2. (b) Yes; 0 .

(c) No

3. (a) Both are 0 .

(b) No; the theorem applies when the partials are taken with respect to the same variables the same number of times.

4. $d^2y/dt^2 = [\sin\phi \cos\theta(d\theta/dt) + \cos\phi \sin\theta(d\phi/dt)](d\rho/dt) +$
$[\sin\theta \cos\phi(d\rho/dt) + \rho\cos\theta\cos\phi(d\theta/dt) - \rho\sin\theta\sin\phi(d\phi/dt)](d\phi/dt) +$
$[\sin\phi\cos\theta(d\rho/dt) + \rho\cos\phi\cos\theta(d\phi/dt) - \rho\sin\phi\sin\theta(d\theta/dt)](d\theta/dt) +$
$\sin\phi\sin\theta(d^2\rho/dt^2) + \rho\sin\theta\cos\phi(d^2\phi/dt^2) + \rho\sin\phi\cos\theta(d^2\theta/dt^2)$

5. (a) $\lim_{\Delta s\to 0} \{[f(r,s+\Delta s,p) - f(r,s,p)]/\Delta s\}$

(b) If f is defined in a disk about (x_0,y_0), then f is continuous at (x_0,y_0) provided $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

6. $AB = \begin{bmatrix} 17 & 6 & 11 \\ 6 & 5 & 8 \\ 11 & 8 & 13 \end{bmatrix}$; $BA = \begin{bmatrix} 14 & 12 \\ 12 & 21 \end{bmatrix}$.

7. (a) $192/125 = 1.536$

(b) $k = 2.5 + (r - 5)/2 + 5(m - 1) - 5(p - 2)/4$

8. (a) $\underline{i} - \underline{j} - \underline{k}$

(b) $\lim_{y\to 0} f(0,y) = -1$ and $\lim_{x\to 0} f(x,0) = +1$.

9. (a) $1/s$

(b) $1/2z$

(c) $k\sin(gb) + gbk\cos(gb)$

10. (a) $350/3$

(b) The number of new pimples for each pound of fatty foods eaten in the previous month.

(c) $dz/dh = (\partial z/\partial f)(df/dh) + (\partial z/\partial c)(dc/dh) + (\partial z/\partial w)(dw/dh)$

(d) There would be infinitely many pimples on the face.

COMPREHENSIVE TEST FOR CHAPTERS 13 - 15 (Time limit: 3 hours)

1. True or false. If false, explain why.

   (a) The derivative of a vector function is <u>always</u> another vector function.

   (b) For matrices A and B, AB never equals BA.

   (c) If x, y, and z are parametrized by linear functions, the curve in space described by x, y, and z is a straight line.

   (d) The value of a 3×3 determinant may not equal the value of a 2×2 determinant.

   (e) The dot product of a vector in the plane and a vector in space does not exist, i.e., $(3\underline{i} + \underline{j})\cdot(\underline{i} + \underline{j} + \underline{k})$ does not exist.

   (f) $x^2 + y^2 + z^2 = 1$ for $z \geqslant 0$ is a function of x and y.

   (g) The unit hemisphere for $x \geqslant 0$ is the graph of a function of x and y.

   (h) If $f(x,y) = 3x^2 + y^2$, then all level curves, except for $c = 0$, are ellipses.

   (i) All equations of the form $f(x,y) = ay + b$ represent straight lines in space.

   (j) If $\underline{u}$ and $\underline{v}$ are vector functions of t, then $(d/dt)(\underline{u}/\underline{v}) = (\underline{u}'\cdot\underline{v} - \underline{u}\cdot\underline{v}')/(\underline{v}\cdot\underline{v})$.

2. Multiple choice.

   (a) If $g(p,s)$ is a function of two variables, then $\partial^2 g/\partial p\partial s = \partial^2 g/\partial s\partial p$ unless:

   (i) $\partial g/\partial s = \partial g/\partial p = 0$ at (s_0,p_0).

   (ii) the third partial derivatives are not continuous.

   (iii) the second partial derivatives are not continuous.

   (iv) none of the above; it is always true.

2. (b) The spherical coordinates of a point are $(4,-3\pi/4,2\pi/3)$. The corresponding cylindrical coordinates are:

(i) $(-\sqrt{12},3\pi/4,-2)$.

(ii) $(-\sqrt{6},-\sqrt{6},-2)$.

(iii) $(\sqrt{12},-3\pi/4,-2)$.

(iv) none of the above.

(c) The graph of $2x^2 + 5xy + y^2 = 0$ is:

(i) a translated circle.

(ii) a rotated ellipse.

(iii) a rotated hyperbola.

(iv) a translated ellipse.

(d) The equation of the line containing $(-1,0,3)$ and pointing in the direction of $2\underline{i} - \underline{j} + \underline{k}$ is:

(i) $(x,y,z) = (-1,0,3) + (2,-1,1)t$.

(ii) $(x+1)/2 = -y = z - 3 = t$.

(iii) both of the above.

(iv) none of the above.

(e) If $f(x,y) = \exp(xy) + 3$, then $\partial f/\partial x$ is:

(i) $x \exp(xy)$.

(ii) $y \exp(xy)$.

(iii) $x \exp(xy) + y \exp(xy)$

(iv) $ye^{xy}\underline{i} + xe^{xy}\underline{j}$.

3. Differentiation problems.

(a) If $\underline{v}(t) = (t^2 - t)\underline{i} + [(5 - t)/(t^2 - 1)]\underline{j} + t\underline{k}$, what is $(d/dp)\underline{v}(pe^{-p})$?

(b) If $\underline{u}(t) = (t^3 - t^2 + 1)\underline{i} - (5 \ln t)\underline{j}$ and $\underline{v}(t) = 3te^t\underline{i} - (6 \cos t - 4t)\underline{j} + (5 \sin(t/2))\underline{k}$, what is $(d/dt)[\underline{u}(t) \times \underline{v}(t)]$? Do not simplify.

3. (c) Let $x = u^2 - v^2$, $y = u^2 - 3uv$, $u = re^s$, and $v = \sin(rs)$.

Write $\partial(x,y)/\partial(r,s)$ as a product of matrices.

4. A surface has the equation $x^2 + 4y^2 - z^2 = 4$.

(a) Describe the level curves for $z = C$, a constant.

(b) Sketch the surface.

(c) At the point $(1,1,1)$, is z changing more rapidly as x changes or more rapidly as y changes? Explain.

5. Let $A = (3,5,7)$, $B = (-1,0,-1)$, and $C = (0,6,2)$ be three points in space.

(a) Find the area of the triangle with vertices at A, B, and C.

(b) For the triangle in part (a), what is $\cos(\angle B)$?

(c) Find the equation of the plane containing A, B, and C.

(d) Compute a unit normal vector to the plane in part (c).

6. At the carnival, a bean bag is thrown at a human target. Its position vector is $-20t\underline{i}$; however, fans are displacing the bean bag's position by the vector $3\underline{i} + t\underline{j} - t\underline{k}$.

(a) What would be the bean bag's position vector if the fans stopped blowing?

(b) How far would the bean bag travel from $t = 0$ to $t = 1$ if the fans had stopped blowing?

(c) What is the curvature of the bean bag's path in still air at $t = 1$?

7. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & -1 & 0 \end{bmatrix}$.

(a) Compute $\det[A]$.

(b) Interpret your answer in part (a) geometrically. Your answer should involve all of the entries of A.

(c) If a 3×3 determinant is zero, what can you say about the geometry?

8. Let $\underline{u} = -\underline{i} - \underline{j}$ and $\underline{v} = \underline{i} + \underline{j} + \underline{k}$.

   (a) Sketch $2\underline{u}$, $\underline{v}$, and $2\underline{u} - \underline{v}$ on the same graph.

   (b) Compute the orthogonal projection of $\underline{u}$ on $2\underline{v}$.

   (c) Find a unit vector $\underline{p}$, which is orthogonal to both $\underline{u}$ and $\underline{v}$, and such that $\underline{u}$, $\underline{v}$, $\underline{p}$ form a left-handed system.

9. Miscellaneous questions.

   (a) For 2×2 matrices A and B , show that $\det(AB) = (\det A) \times (\det B)$.

   (b) Find the distance from the point $(1,-1,2)$ to the line passing through $(5,-3,0)$ and $(7,6,-1)$.

   (c) Use the linear approximation to estimate $f(x,y,z) = xy + 2xz + 3yz$ at the point $(-1,0.97,2.05)$.

10. Practical application.

    A drunk driver has to serve three days in jail. The jailbird consumes q_{b1} slices of bread on day 1, q_{b2} slices on day 2, and q_{b3} slices on day 3. He drinks q_{w1} cups of water on day 1, q_{w2} on day 2, and q_{w3} on day 3. The price of bread per slice is p_b . The price of water per cup is p_w .

    (a) What does each entry of the product $[p_b \quad p_w]\begin{bmatrix} q_{b1} & q_{b2} & q_{b3} \\ q_{w1} & q_{w2} & q_{w3} \end{bmatrix}$ tell you?

    (b) If the product in part (a) is AB and you wanted to compute ABC , would you choose C to be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ or $[1 \quad 1 \quad 1]$?

    (c) By performing the multiplication described in part (b), what does your answer tell you?

ANSWERS TO COMPREHENSIVE TEST

1. (a) True

(b) False; let $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) True

(d) False; consider $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$.

(e) False; the vector $a\underline{i} + b\underline{j}$ is the same as $a\underline{i} + b\underline{j} + 0\underline{k}$.

(f) True

(g) False; for every (x,y) in the domain, there are two corresponding z values except on the xy-plane.

(h) True

(i) False; they are planes.

(j) False; division of vectors is undefined.

2. (a) iii

(b) iii

(c) iii

(d) iii

(e) ii

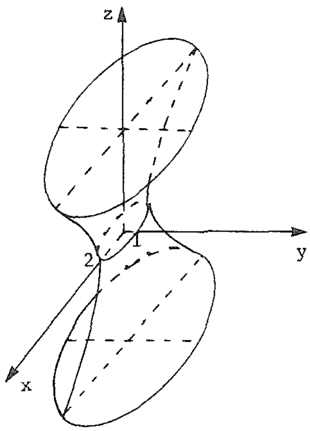
3. (a) $(1 - p)e^{-p}\{(2pe^{-p} - 1)\underline{i} + [(p^2e^{-2p} - 10pe^{-p} + 1)/(p^2e^{-2p} - 1)^2]\underline{j} + \underline{k}\}$

(b) $[(3t^2 - 2t)\underline{i} - (5/t)\underline{j}] \times [3te^t\underline{i} - (6 \cos t - 4t)\underline{j} + (5 \sin(t/2))\underline{k}] +$
$[(t^3 - t^2 + 1)\underline{i} - (5 \ln t)\underline{j}] \times [(3 + 3t)e^t\underline{i} - (6 \sin t - 4)\underline{j} +$
$((5/2)\cos(t/2))\underline{k}]$

(c) $\begin{bmatrix} 2u & -2v \\ 2u - 3v & -3u \end{bmatrix} \begin{bmatrix} e^s & re^s \\ s\cos(rs) & r\cos(rs) \end{bmatrix}$

4. (a) Ellipses with minor axis parallel to the y-axis having length $\sqrt{4 + c^2}$ and with major axis parallel to the x-axis having length $2\sqrt{4 + c^2}$.

(b)



(c) As y changes; because at $(1,1,1)$, $\partial z/\partial x = 1$ and $\partial z/\partial y = 4$.

5. (a) $\sqrt{1466}/2$

(b) $58/\sqrt{4830}$

(c) $-33x - 4y + 19z - 14 = 0$

(d) $(-33\underline{i} - 4\underline{j} + 19\underline{k})/\sqrt{1466}$

6. (a) $(-20t - 3)\underline{i} - t\underline{j} + t\underline{k}$

(b) $\sqrt{402}$

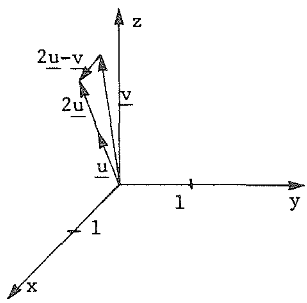
(c) $\sqrt{2}/531\sqrt{59}$

7. (a) -22

(b) The volume of the parallelepiped spanned by the vectors $\underline{i} + 2\underline{j} + 3\underline{k}$, $2\underline{i} + \underline{j} - \underline{k}$, and $3\underline{i} - \underline{j}$ is 22.

(c) The three vectors lie in a plane or on a line.

8. (a)



(b) $(2/3)(\underline{i} + \underline{j} + \underline{k})$

(c) $(\underline{i} - \underline{j})/\sqrt{2}$

9. (a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Both sides equal adeh + bcgf - adgf - bceh .

(b) $\sqrt{1000/43}$

(c) 0.90

10. (a) How much it costs to feed bread and water to the prisoner each day.

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) The total cost of feeding bread and water to the prisoner during his prison term.